Fractionalized quantum spin Hall effect

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Effects of electron correlations on a two-dimensional quantum spin Hall (QSH) system are studied. We examine possible phases of a generalized Hubbard model on a bilayer honeycomb lattice with a spin-orbit coupling and short-range electron-electron repulsions at half filling, based on the slave-rotor mean-field theory. Besides the conventional QSH phase and a broken-symmetry insulating phase, we find a third phase, a fractionalized quantum spin Hall phase, where the QSH effect arises for fractionalized spinons which carry only spin but not charge. Experimental manifestations of the exotic phase and effects of fluctuations beyond the saddle-point approximation are also discussed.

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I. INTRODUCTION

The quantum spin Hall (QSH) phase is a state of matter which arises due to spin-orbit coupling in time-reversal symmetric systems.[1,2] It is characterized by a gap in the bulk and an odd number of Kramers pairs of gapless edge modes which are protected by a Z_2 topological order.[1–3] Recently, an experimental signature for the gapless edge modes has been observed in HgTe quantum wells.[4] Although the QSH state was proposed in a noninteracting system,[1] the gapless edge modes are stable in the presence of weak time-reversal symmetric disorder or many-body interactions,[5,6] which suggests that the topological order in the bulk is also robust against weak disorder[7] and interactions.[8–10] If electron correlations become sufficiently strong, a broken-symmetry insulating phase can be stabilized. Recently, a possibility of the QSH effect arising from many-body interactions has also been studied.[11,12]

Fractionalized phases are another unconventional state of matter and arise as a result of electron correlations. In the absence of spin-orbit coupling, a subtle balance between the kinetic energy of electrons and electron-electron interactions can stabilize spin liquid phases where spins remain disordered due to quantum fluctuations while charge excitations are gapped.[13,14] Spin liquids exhibit fractionalization in that the low energy excitations are spinons which carry only spin but not charge. Much attention has been paid to two-dimensional frustrated magnets which are candidates for spin liquid states.[15]

Since either spin-orbit couplings or electron correlations can lead to an interesting phase, what happens when both of these interactions are important? We address this question by examining the possibility of a new phase of matter arising due to an interplay between the spin-orbit coupling and electron correlations. A *fractional* QSH state which corresponds to a time-reversal symmetric version of the fractional quantum Hall state has been suggested as a possible phase for interacting systems with spin-orbit coupling.[2] In this paper, we explore an alternative possibility where the QSH effect arises simultaneously with *fractionalization* in a spin liquid state. The honeycomb lattice is an ideal geometry to study such effects because it may support both the QSH phase[1] and the spin liquid phase.[16,17]

II. MODEL

We consider a generalized Hubbard model defined on a double layer of honeycomb lattice

$$H = -\sum_{\substack{(i,j)\\ a,\sigma}} (t_{ija\sigma} c^{\dagger}_{ia\sigma} c_{ja\sigma} + \text{H.c.}) + U \sum_{i,a} (n_{ia}-1)^2 + U' \sum_{i} (n_{i1}-1)(n_{i2}-1) - \sum_{ia} \mu_a (n_{ia}-1), \qquad (1)$$

where $c^{\dagger}_{ia\sigma}$ is the creation operator for an electron of spin $\sigma = \pm 1$ on site i of layer $a=1$ or 2 and n_{ia} is the number operator. U (U') is the on-site (interlayer) Coulomb repulsion and μ_a is the chemical potential which is tuned so that each layer is at half filling. The intralayer tunneling amplitudes are $t_{ija\sigma}=t$ when (i,j) are nearest-neighbor (NN) sites and $t_{ija\sigma}=\delta_{a1} t' e^{i\phi_{ij}\sigma}$ for next-nearest-neighbor (NNN) sites. We assume that there is no spin-orbit coupling or NNN hopping in the second layer and no interlayer tunneling. The spin-dependent phase $\phi_{ij}\sigma$, which we take to be positive (negative) if an electron with spin up (down) hops around the lattice in a counterclockwise sense, is due to spin-orbit coupling.[1] We emphasize that our model is an idealized model and the goal of our investigation is to demonstrate the possibility of finding a new state of matter from a simple model which contains both spin-orbit coupling and electron correlations.

We now represent the Hamiltonian in the slave-rotor representation[18] $c_{ia\sigma}=e^{-i\theta_{ia}} f_{ia\sigma}$, where the spinon operator $f_{ia\sigma}$ carries only spin and the chargon operator $e^{i\theta_{ia}}$ carries only charge. The enlarged Hilbert space is constrained by $L_{ia}=\Sigma_{\sigma} f^{\dagger}_{ia\sigma} f_{ia\sigma}-1$, where $L_{ia}=n_{ia}-1$ represents the charge quantum number, conjugate to θ_{ia}. Integrating out L_{ia} (see Appendix A), we obtain the partition function, $\mathcal{Z} = \int Df^{*} Df D\theta Dh e^{-\int d\tau L}$, where the Euclidean Lagrangian is given by

$$L = \sum_{\substack{i,a\\ \sigma}} f^*_{ia\sigma}\partial_\tau f_{ia\sigma} + \sum_{i,a} (ih_{ia} - \mu_a)\left(\sum_\sigma f^*_{ia\sigma} f_{ia\sigma} - 1\right) - \sum_{\substack{(i,j)\\ a,\sigma}} (t_{ija\sigma} f^*_{ia\sigma} f_{ja\sigma} e^{i(\theta_{ia}-\theta_{ja})} + \text{H.c.}) + \frac{1}{U_+}\sum_i (\partial_\tau\theta_{i+} + h_{i+})^2 + \frac{1}{U_-}\sum_i (\partial_\tau\theta_{i-} + h_{i-})^2. \tag{2}$$

Here $U_\pm \equiv 2U \pm U'$ and $A_\pm \equiv (A_1 \pm A_2)/2$ for $A_a = \theta_{ia}$ or h_{ia}, where h_{ia} is a Lagrange multiplier field enforcing the constraint. In this paper, we concentrate on the parameter region $U_- \ll U_+, t$, in which case the phase stiffness for the θ_{i-} field is large and the phases of the antisymmetric chargon field in the two layers are locked together. If θ_{i-} is condensed, both θ_{i-} and h_{i-} are gapped due to the Higgs mechanism. At low energies, we can set the chargon fields and the Lagrange multipliers in the two layers equal to each other and our model reduces to a model with one chargon field θ_i and one Lagrange multiplier h_i. When $U_- = 0$, this effective model for the single chargon becomes exact (see Appendix B).

We now decouple the quartic terms in the hopping sector by a Hubbard-Stratonovich transformation to obtain the effective Lagrangian

$$\begin{aligned} L = {} & \sum_{\langle i,j\rangle} t[\chi^{f^*}_{ij}\chi^X_{ij} + \text{H.c.}] + \sum_{\langle\langle i,j\rangle\rangle} t'[\chi^{f'^*}_{ij}\chi^{X'}_{ij} + \text{H.c.}] \\ & + \sum_{i,a}\mu_a + \sum_i \left(\bar{\lambda}_i + 2\bar{h}_i\right) + \sum_{i,a,\sigma} f^*_{ia\sigma}(\partial_\tau - \bar{h}_{ia} - \mu_a) f_{ia\sigma} \\ & - \sum_{\substack{(i,j)\\ a,\sigma}} t[\chi^X_{ij} f^*_{ia\sigma} f_{ja\sigma} + \text{H.c.}] \\ & - \sum_{\substack{\langle\langle i,j\rangle\rangle\\ \sigma}} t'[\chi^{X'}_{ij} e^{i\phi_{ij}\sigma} f^*_{i1\sigma} f_{j1\sigma} + \text{H.c.}] \\ & + \frac{1}{U_+}\sum_i (\partial_\tau + \bar{h}_i) X^*_i (\partial_\tau - \bar{h}_i) X_i - \sum_{\langle i,j\rangle} t[\chi^f_{ij} X^*_i X_j + \text{H.c.}] \\ & - \sum_{\langle\langle i,j\rangle\rangle} t'[\chi^{f'}_{ij} X^*_i X_j + \text{H . c.}] - \sum_i \bar{\lambda}_i |X_i|^2. \end{aligned} \tag{3}$$

Here a soft boson field $X_i \equiv e^{-i\theta_i}$ has been introduced with a Lagrange multiplier λ_i which imposes the constraint $|X_i| = 1$. $\bar{\lambda}_i = -i\lambda_i$ and $\bar{h}_i = -ih_i$ are the saddle-point values of the Lagrange multipliers and lie on the imaginary axis.[16] χ^f_{ij} and χ^X_{ij} ($\chi^{f'}_{ij}$ and $\chi^{X'}_{ij}$) are the NN (NNN) hopping order parameters of spinon and chargon, respectively.

III. MEAN-FIELD PHASE DIAGRAM

In the small U limit, the system essentially reduces to a noninteracting model with no coupling between the layers. In this limit, the conventional QSH phase will be realized in the first layer where there is spin-orbit coupling. In the second layer, the semimetal (SM) phase with gapless Dirac fermions will be obtained.

When $U \gg t, t'$, the Coulomb interactions are dominant and the low energy states of the system are described by the configurations which satisfy $\Sigma_a n_{ia} = 2$. To second order in t and t', the low energy effective Hamiltonian is obtained to be

$$H_{\text{eff}} = \frac{t^2}{U}\sum_{\langle i,j\rangle} \text{tr}[\mathcal{Q}_i\mathcal{Q}_j] + \frac{t'^2}{U}\sum_{\langle\langle i,j\rangle\rangle} \text{tr}[e^{-i\phi_{ij}\sigma^3}\mathcal{T}_i e^{i\phi_{ij}\sigma^3}\mathcal{T}_j], \tag{4}$$

where $[\mathcal{Q}_i]_{a\sigma,b\sigma'} = c^\dagger_{ia\sigma} c_{ib\sigma'}$ is the 4×4 matrix of $U(4)$ generators and $[\mathcal{T}_i]_{\sigma,\sigma'} = c^\dagger_{i1\sigma} c_{i1\sigma'}$ is the 2×2 matrix of $U(2)$ generators which are restricted to the first layer. The first term has a $U(4) = U(1)\otimes SU(4)$ symmetry where the $U(1)$ is associated with conservation of the total charge and the $SU(4)$ with conservation of the flavor quantum number given by the layer index and the spin. The six states which satisfy the constraint $\Sigma_a n_{ia} = 2$ at each site form the rank 2 antisymmetric representation of the $SU(4)$ group. The second term breaks the $U(4)$ symmetry into $SU(2)\otimes U(1)^3$ where the unbroken $SU(2)$ symmetry is the spin-rotational symmetry in the second layer and the three $U(1)$ symmetries are associated with charge conservation in each layer and S_z conservation in the first layer. If $t' = 0$, each NN bond tends to form an $SU(4)$ singlet and a valence bond solid (VBS) phase which breaks translational symmetry is a good candidate for the ground state.[20] The most natural pattern among possible VBS states in the honeycomb lattice is the dimerized phase where valence bonds are stronger for the bonds which are directed along one of the six symmetry directions.[16] A nonzero t' will enhance quantum fluctuations, but we expect that the fully gapped dimerized state will remain stable for a finite range of $t' < t$.

With the guidance of these insights, the mean-field theory is carried out for the uniform and dimer ansatze. We solve a system of self-consistent equations at $T = 0$ for the link order parameters, chemical potentials, and Lagrange multipliers by requiring that the energy remains stationary with respect to variations of those variables. We then find the mean-field phase diagram by choosing the lower energy configuration between the dimer and uniform ansatze. In particular, we are interested in finding a new phase in the insulating side of the phase diagram where both the spin-orbit coupling and the electron correlation are important. Although we could start from an effective "spin" model to study such insulating phases, we will use the full action in Eq. (3) which is applicable in all parameter regimes. It would be of interest to study the possibility of obtaining an exotic phase in an effective model such as Eq. (4), possibly with additions of higher order ring-exchange terms.

For large U, we find that the dimerized configuration has lower energy, while for small U the uniform configuration has lower energy, as expected. There is a first-order phase transition between these two phases. Within the uniform phase, an onset of the chargon condensation marks another phase transition. Although not shown here, the chargon gap vanishes continuously as U decreases and the phase transition is a second-order phase transition. The Bose condensation amplitude is given by $Z = |\langle X\rangle|^2$. If the chargons are condensed, a spinon recombines with a chargon to become an electron. This phase is the conventional weakly interacting phase where the electrons form the QSH phase in the first layer while the semimetal phase with Dirac nodes is realized

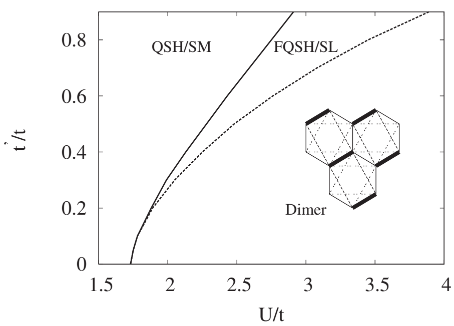

FIG. 1. Phase diagram in the space of t'/t and U/t in a 40×40 lattice with $U_{-}=0$. The weakly interacting phase (small U) has $Z \neq 0$ and the first layer forms the conventional QSH phase while the second layer is in the semimetal (SM) phase with gapless Dirac nodes. The intermediate region has the fractionalized quantum spin Hall (FQSH) phase with $Z=0$ where chargeless spinons form the QSH phase in the first layer and the gapless spin liquid (SL) phase in the second layer. In both QSH/SM and FQSH/SL phases, the NN and NNN hopping order parameters are nonzero and site independent. The large U region is a dimerized phase where $Z=0$ and the hopping order parameters along the bold lines have the maximum amplitude and all other bonds have zero amplitude. The solid line represents the second-order transition and the dotted line the first-order transition.

in the second layer. For $t'/t<0.2$, the first-order phase transition from the uniform phase to the dimerized phase occurs before the Bose condensation amplitude Z in the uniform phase becomes zero as U/t increases so that there is no intermediate phase between the conventional QSH/SM phase and the dimerized phase. On the other hand, for $t'/t>0.2$, a window opens up for an intermediate phase and the region of stability for the intermediate phase grows as t' is increased. The mean-field phase diagram is shown in Fig. 1.

The intermediate phase is characterized by the uniform link order parameters but, unlike the QSH/SM phase, the chargons remain gapped, which makes it an insulating phase. This is a phase where the fractionalized spinon arises as a low energy excitation. In the first layer, the spinon is gapped in the bulk due to the spin-dependent phase in the spinon hopping which has been inherited from the spin-orbit coupling of electrons as is shown in the seventh term of Eq. (3). At the mean-field level, which ignores the fluctuations of the order parameters, the spinon spectrum is essentially the same as the electron spectrum in the Kane and Mele model.[1] The nontrivial topological structure in the spectrum guarantees that there exist gapless edge modes in the first layer. Therefore, we have a fractionalized quantum spin Hall (FQSH) phase in which the gapless edge states are carried by spinons and not by electrons as in the conventional QSH phase. It is noted that the gapless edge mode and the FQSH state may be robust even though S_z symmetry is broken in the first layer as will be discussed in Sec. IV. In the second layer, the spins form an algebraic spin liquid (SL),[21] whose low energy excitations are described by four two-component Dirac spinons.

The electromagnetic response and transport properties of the FQSH phase are very different from those of the usual QSH phase, as discussed below in Sec. V. It was recently pointed out that the conventional QSH state can have spin-charge separated excitations in the presence of π flux even in the absence of many-body interactions.[9] We emphasize that the spinon which arises in the FQSH phase is different in that they are intrinsic excitations resulting from many-body correlations while the fractionalized excitations obtained in the noninteracting systems are generated by an external fractional magnetic-flux quantum. The QSH effect in the presence of a Z_2 gauge field was recently studied where the dynamic fluxon makes the fractionalized excitation a propagating mode.[10]

IV. STABILITY OF THE EDGE MODES

Beyond the mean-field approximation, the most important fluctuations are the phase fluctuations of the hopping order parameters. The phase mode is described by a gauge field because it restores the gauge invariance associated with the local phase transformation $f_{ia\sigma} \to e^{i\varphi_i} f_{ia\sigma}$ and $\theta_i \to \theta_i + \varphi_i$. The low energy effective theory in the FQSH/SL phase is given by

$$S=\sum_{n,\sigma}\int d\tau dx_1 dx_2 \bar{\psi}_{n\sigma}(i\gamma^{\mu}D_{\mu})\psi_{n\sigma} + \frac{1}{g^2}\int d\tau dx_1 dx_2 f_{\mu\nu}f_{\mu\nu} + \int d\tau dx_1 \bar{\eta}(i\gamma^{a}D_{a})\eta. \quad (5)$$

Here $\psi_{n\sigma}$ is the 2+1D massless Dirac fermion in the second layer, σ labels spin, and $n=1,2$ is the index for the nodal points. $D_{\mu}=\partial_{\mu}-ia_{\mu}$ is the covariant derivative, a_{μ} is the internal gauge field, and $f_{\mu\nu}$ is the field strength tensor with $\mu=0,1,2$. η is the 1+1D Dirac fermion on the edge of the first layer with $a=0,1$. The edge is assumed to be along the x_1 direction. Although the gauge field is a compact $U(1)$ gauge field, the compactness is unimportant at low energies when S_z is conserved[22] or a large number of gapless Dirac fermions are coupled with the gauge field.[23] In our case, there are $N=4$ gapless Dirac fermions coming from the second layer. In the following, we proceed with the assumption that the four gapless Dirac fermions are enough to stabilize the fractionalized phase against proliferation of instantons. It is noted that the stability of the FQSH state relies on the existence of both layers. The spin-dependent NNN hopping in the first layer opens up the gap of the spinon in the first layer which provides the robustness of the edge modes. The presence of the second layer is crucial in that the gapless spinons screen the gauge field and suppress the gauge fluctuations.

The $U(1)$ gauge field is coupled to the spinons in both layers. The spinons are gapped in the bulk of the first layer but there are gapless edge modes. Although the existence of the gapless modes has been inferred from the mean-field band structure which has a nontrivial topological order, the stability of those edge modes is less clear in this case because they are coupled to the gapless gauge field. The key question is whether the fluctuating gauge field destabilizes the topological order associated with the spinon band to open up a gap for the edge modes. In order to address this question, one can integrate out the bulk degrees of freedom in Eq. (5) to obtain an effective action at the edge. The resulting theory is an $1+1$D quantum electrodynamics (QED) with a nonlocal action for the gauge field. Whatever this nonlocal action is, in the $1+1$D QED the quantum fluctuations of the gapless fermions open up a gap for the gauge field.[24] This suppresses the fluctuations of the gauge field at the edge although the gauge field remains gapless in the bulk.

One may worry about the possibility of direct spin-spin interactions between the two layers destabilizing the edge modes. To examine the stability of the edge modes, one has to consider all the gapless modes in the low energy theory [Eq. (5)]. Since there is no tunneling between the two layers, the lowest-order interlayer interactions that one can add are two-body terms of the form

$$V\int d\tau dx_1 \bar{\psi}(\tau,x_1,x_2=0)\psi(\tau,x_1,x_2=0)\bar{\eta}(\tau,x_1)\eta(\tau,x_1). \tag{6}$$

Since the edge modes in the first layer can only interact locally with the bulk modes in the second layer the integration measure only has one spatial and one temporal component. Neglecting gauge fluctuations and forward scatterings of the edge modes, the free low energy theory is invariant under a scale transformation $(\tau,x_1,x_2)=b(\tau',x_1',x_2')$, $\psi=b^{-1}\psi'$, and $\eta=b^{-1/2}\eta'$ with $b>1$. The interlayer interaction scales as $V'=b^{-1}V$. If we include gauge fluctuations and forward scatterings of the edge modes, the edge mode is described by the Luttinger liquid with a nontrivial Luttinger parameter $K\neq 1$ and the spinons in the second layer are described by the algebraic spin liquid. As a result, the scaling dimension of the interlayer coupling will receive loop corrections which are of the order of $[V]=-1+O(1/N)+O(K-1)$. Given that $N=4$, the interlayer coupling may remain irrelevant if the forward scattering is sufficiently weak. If the interlayer coupling is irrelevant, the edge modes are stable.

V. PHYSICAL PROPERTIES AND DISCUSSION

Now we discuss physical manifestations of the FQSH state. The longitudinal transport properties along the edge are very different from those of QSH states or trivial insulators. There will be a metallic thermal conductivity along the edge due to the gapless edge mode. However, there will be no charge conductivity because the spinon is charge neutral, which is the signature of the spin-charge separation.

The most stark difference from the conventional QSH state lies in the transverse spin transport induced by an external electromagnetic (EM) field. We put the system on a cylinder with two edges at the ends of the cylinder. In the usual QSH state with S_z conservation, upon threading a magnetic-flux quantum through the halo of the cylinder, a spin-up electron is transported from one edge to the other while a spin-down electron is transported in the opposite direction. This results in a transport of net spin $S=1$ from one edge to the other. This is illustrated in Fig. 2(a). On the other hand, in the FQSH phase, the edge modes are neutral spinons which are not directly coupled to the external EM field and there will be no such transverse spin transport. Although spinons are indirectly coupled to external EM fields through chargons, which are coupled to both the external and internal gauge fields, the weak coupling cannot produce a nonzero spin Hall transport because of a nontrivial quantum order associated with the fractionalization. In the fractionalized phase, the tunneling rate of the internal gauge flux from one value to another value is exponentially suppressed with increasing system size and the flux through the cylinder is precisely conserved at $T=0$ in the thermodynamic limit. The internal gauge flux remains strictly at zero under the adiabatic insertion of the flux. Therefore, the external flux does not induce any transverse spin transport, as is illustrated in Fig. 2(b), in sharp contrast to the QSH state. This insensitivity of the edge modes to EM fields can potentially be useful in stabilizing the edge modes in an environment with fluctuating EM fields which induce back scatterings between the edge modes in QSH states.

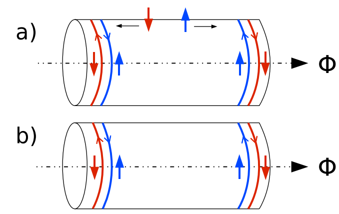

FIG. 2. (Color online) (a) Transverse spin response to an applied external magnetic field in the conventional quantum spin Hall phase. Upon threading a magnetic-flux quantum, a spin up propagates from one edge 1, say, to edge 2 and a spin down propagates from edge 2 to edge 1. (b) The response in the fractionalized quantum spin Hall phase. The external flux does not generate any transverse spin transport because the edge modes are charge neutral spinons.

In summary, we proposed and studied a simple model which has both spin-orbit coupling and many-body interactions. We found a region of the mean-field phase diagram where a fractionalized quantum spin Hall state is stable and argued that this state may survive the effects of fluctuations under certain conditions. In the FQSH state, charge neutral spinons form gapless edge modes which carry only spin, unlike the conventional QSH state where the edge modes carry both charge and spin. Due to the charge neutral edge modes, the FQSH state shows a set of unique transport properties and electromagnetic responses which are distinct from conventional states.

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APPENDIX A: DERIVATION OF THE BOSON ACTION IN EQ. (2)

Since the constraint $L_{ia}=n_{ia}-1$ is diagonal in site indices, calculating its partition function can be reduced to calculating one site matrix elements of the form

$$\mathcal{Z}_{i\theta} = \langle \theta_1' \theta_2' | e^{-\epsilon H_L} | \theta_1 \theta_2 \rangle, \tag{A1}$$

where 1 and 2 refer to the layer index. Here $H_L = U(L_1^2 + L_2^2) + U' L_1 L_2 - i(h_1 L_1 + h_2 L_2)$. The one site partition function becomes

$$\mathcal{Z}_{i\theta} = \sum_{l_1, l_2} e^{i l_1(\theta_1' - \theta_1) + i l_2(\theta_2' - \theta_2) - \epsilon[U(l_1^2 + l_2^2) + U' l_1 l_2 - i(h_1 l_1 + h_2 l_2)]}, \tag{A2}$$

where we have omitted the site dependence of the eigenvalues to simplify the forthcoming formulas.

To obtain the effective action of the θ_{ia} variables we must sum over all l_{ia}. We do this by making a change of variables from the discrete l_{ia} to a new set of "continuous" variables $p_a = \epsilon l_a$. After implementing these changes we then change variables from the original p_1 and p_2 to new symmetric and antisymmetric variables

$$p_\pm \equiv \frac{1}{2}(p_1 \pm p_2), \tag{A3}$$

which allow us to write the one site partition function as two decoupled Gaussian integrals

$$\mathcal{Z}_{i\theta} = \frac{1}{2\epsilon^2} \int dp_+ dp_- \times e^{2ip_+[\dot{\theta}_+ + h_+] + 2ip_-[\dot{\theta}_- + h_-] - 1/\epsilon[(2U+U')p_+^2 + (2U-U')p_-^2]}, \tag{A4}$$

where we have rewritten all fields as symmetric and antisymmetric combinations of the original layer dependent fields and $\dot{\theta}_\pm = (\theta_\pm' - \theta_\pm)/\epsilon$. Defining new coupling constants as $U_\pm = 2U \pm U'$, we obtain

$$\mathcal{Z}_{i\theta} = \frac{1}{2\epsilon^2} \sqrt{\frac{\pi\epsilon}{U_+}} \sqrt{\frac{\pi\epsilon}{U_-}} e^{-\epsilon/U_+(\dot{\theta}_+ + h_+)^2} e^{-\epsilon/U_-(\dot{\theta}_- + h_-)^2}. \tag{A5}$$

The full partition function for the θ variables is obtained by taking a product over all lattice sites of the single site result above. This gives the last two terms in Eq. (2).

APPENDIX B: EXACTNESS OF ONE-BOSON THEORY WHEN $U' = 2U$

In Sec. II we argued that in the region $U' \approx 2U$ our model reduces to the one-boson model through the Higgs mechanism. Here we show that the one-boson model becomes exact when $U' = 2U$. For $U' = 2U$, we can write Hamiltonian (1) as

$$H = -\sum_{\substack{(i,j)\\ a,\sigma}} (t_{ija\sigma} c_{ia\sigma}^\dagger c_{ja\sigma} + \text{H.c.}) + U\left(\sum_{\alpha=1}^{4} c_{i\alpha}^\dagger c_{i\alpha} - 2\right)^2 - \sum_{ia} \mu_a (n_{ia} - 1), \tag{B1}$$

where we have introduced an $SU(4)$ index $\alpha = 1, \ldots, 4$ defined as $1 = (1\uparrow)$, $2 = (1\downarrow)$, $3 = (2\uparrow)$, and $4 = (2\downarrow)$; the first letter in the parentheses is the layer index and the arrows represent the eigenvalue of S_z. The Coulomb term is now an $SU(4)$ symmetric interaction term.

We can now decompose the electron operator into a spinon part and a chargon part as $c_{i\alpha} = f_{i\alpha} e^{-i\theta_i}$, where the $SU(4)$ quantum number is carried by the spinon. With this decomposition we obtain the slave-rotor representation for an $SU(4)$ model[19]

$$H = -\sum_{\substack{(i,j)\\ a,\sigma}} (t_{ij\alpha} f_{i\alpha}^\dagger f_{j\alpha} e^{i(\theta_i - \theta_j)} + \text{H.c.}) + U \sum_i L_i^2 + i \sum_i h_i \left(\sum_\alpha f_{i\alpha}^\dagger f_{i\alpha} - L_i - 2\right) - \sum_{i,\alpha} \tilde{\mu}_\alpha \left(f_{i\alpha}^\dagger f_{i\alpha} - \frac{1}{2}\right). \tag{B2}$$

Here h_i is the Lagrange multiplier which imposes the constraint $L_i = \sum_\alpha f_{i\alpha}^\dagger f_{i\alpha} - 2$ with L_i being the conjugate variable to θ_i. We have defined a new chemical potential $\tilde{\mu}_\alpha = \mu_1$ if $\alpha = 1, 2$ and $\tilde{\mu}_\alpha = \mu_2$ if $\alpha = 3, 4$. If we apply the similar Hubbard-Stratonovich transformation to this Hamiltonian we would reproduce the effective action in Eq. (3).

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